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Supersymmetric solutions to gauged $N = 2\ d = 4$ SUGRA: the full timelike shebang

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Abstract

We discuss the structure of supersymmetric solutions in the timelike case to general gauged $N = 2\ d = 4$ supergravity theories coupled to non-Abelian vector multiplets and hypermultiplets.

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Introduction

The starting shot for the systematic characterization of supersymmetric solutions to supergravity theories was given in 1982 by Gibbons & Hull [1], who obtained a partial characterization of the supersymmetric solutions of pure (minimal) $N = 2$ $d = 4$ supergravity, later completed by Tod [2] using the consistency conditions of the Killing spinor equations, who realized that the assumption of hypesurface orthogonality implicitly made by Gibbons and Hull was unnecessary for a solution to be supersymmetric. In a related development, in 1985 Kowalski-Glikman [3] proved that the only solutions to minimal $N = 2$ $d = 4$ SUGRA that do not break any supersymmetries, called the maximally supersymmetric solutions, are Minkowski space, the Robinson-Bertotti spacetime ($aDS_2 \times S^2$) and a specific pp-wave called the 4-dimensional Kowalski-Glikman wave.

In the 30 years since, hosts of results[3] concerning supersymmetric solutions to SUGRAs have been obtained and many potent techniques were developed in order to obtain them. The first of such techniques was developed by Gauntlett *et al.* in ref. [4] and used it to give a complete classification of supersymmetric solutions to minimal $N = 1$ $d = 5$ SUGRA; this technique goes by the name of *bilinear method* as it deals with the analysis of all the form-fields one can construct as bilinears out off the Killing spinors. In this method there are 2 types of relations for the bilinears: first of all there are "kinematical" relations between products of bilinears due to the Fierz identities and only depend on the number and type of spinors employed in a given theory, and not on the theory itself (matter content, equations of motion etc.). The second kind of relations are "dynamical" in that they are differential relations determining the spacetime dependency of the bilinears and which originate in the theory-specific Killing spinor equations. In this article we will use the bilinear method, but it must be mentioned that there are more techniques *e.g.* *spinorial geometry* proposed by Gillard

[3] We feel that it is sheer impossible to give an overview doing justice to all the interesting results obtained and shall restrict ourselves mainly to the results concerning the classification of supersymmetric solutions to $N = 2$ $d = 4$ supergravity and apologize in advance for any omission.

et al. in ref. [5], which is extremely powerful and much of the progress in the characterization of supersymmetric solutions, especially to higher dimensional SUGRAs, was made using it.

Another technique that was developed in ref. [4] is an effort-saving one: preserved supersymmetry implies, by means of the integrability condition for the existence of a Killing spinor, relations between the equations motions implying that there is a minimal set of (components of) equations of motion that, once these are satisfied, automatically ensure that all the equations of motion are satisfied. In ref. [6] this effort-saving technique was linked to the so-called *Killing Spinor Identities* originally derived in ref. [7]; the KSIs are the restriction of the gauge identities expressing the fact that a SUGRA action is supersymmetric, to the case of vanishing fermionic fields and with a gauge parameter taken to be the Killing spinor. The bottom line of [6]'s identification is that there is no need to calculate the integrability conditions and that only the supersymmetric variations of the bosonic fields need to be known.

The generalization of Gibbons & Hull's result to vector multiplet-coupled $N = 2$, $d = 4$ SUGRA was done first for static spacetimes in refs. [8] and in ref. [9] for general stationary solutions. In ref. [10] the authors carried out a full characterization of supersymmetric solutions to this theory finding in the timelike case full agreement with aforementioned works; the null case was found to allow not only for pp-waves but also for stringy cosmic strings of the type first studied in [11]. This characterization was then extended to the case of $N = 2$, $d = 4$ SUGRA coupled to vector multiplets and hypermultiplets in ref. [12] and to the case of YM-vector multiplets in refs. [13, 14]; in the latter theories one can, depending on the model, construct analytic, globally regular monopole solutions and non-Abelian black holes. Caldarelli & Klemm [15] extended Tod's results to the case of minimal gauged $N = 2$, $d = 4$ supergravity and the resulting solutions were studied further by Cacciatori *et al.* in refs. [16]; some examples of supersymmetric black holes had been already been obtained in refs. [17, 18]. The fact that the maximally supersymmetric solution to this theory is aDS_4, was established by Kowalski-Glikman in ref. [19] and in refs. [20] it was shown that all solutions preserving more than half of the supersymmetry necessarily arise as quotients of aDS_4. In refs. [21] the characterization of supersymmetric solutions to minimal gauged $N = 2$, $d = 4$ SUGRA was extended by considering the coupling to Abelian vector multiplets and (rotating) black hole solutions were constructed in refs. [22]. Finally, in refs. [23] a classification was made for the fake-SUGRA analogue of gauged minimal $N = 2$, $d = 4$ SUGRA coupled to YM-vector multiplets, leading to generalizations of Kastor & Traschen's cosmological multi-black hole solutions [24]; as these theories are obtained by Wick-rotating the $\mathrm{U}(1)$ Fayet-Iliopoulos term, the potential has the opposite sign w.r.t. supersymmetric theory, the maximally fake-supersymmetric solution is 4-dimensional De Sitter space.

What for the moment is missing from the above laundry list of classification articles is the characterization of supersymmetric solutions to gauged $N = 2$, $d = 4$ supergravity coupled to YM-vector multiplets and hypermultiplets: the aim of the current article is to do just that, albeit for the timelike case only.

Observe that this is no way means that there are no supersymmetric solutions to the full theory known to the literature: for example supersymmetric domain walls were constructed in ref. [25], recently the maximally supersymmetric solutions were classified in ref. [26] and supersymmetric Lifschitz, Schrödinger and (anti-)De Sitter solutions were considered in refs. [27, 28]; Supersymmetric black-hole solutions with an Abelian gauging were constructed in ref. [29] and further analyzed in ref. [30]. Lastly, let us mention refs. [31] in which the Bogomol'nyi bound for asymptotically aDS_4 black holes and black strings are discussed.

The outline of this article follows the algorithm used in the classification of supersymmetric solutions and is as follows: section 1 contains an extremely short introduction to gauged $N = 2$, $d = 4$

SUGRA coupled to YM-vector and hypermultiplets. In section 2 we formulate the basic problem of finding SUGRA solutions preserving some supersymmetry as the problem of finding expressions for the purely bosonic SUGRA fields that allow for supersymmetry transformations that do not generate non-trivial fermionic fields; the relevant equations to be solved are called the Killing Spinor Equations (KSEs) and the supersymmetric variation parameter is called the Killing Spinor. Given that information, we detail the KSIs and discuss the minimal set of (components of) equations that must be checked explicitly as to be sure that all the equations of motion are satisfied. Then in section 3 we analyze the differential constraints on the bilinears[4] in general and in section 4 we shall introduce coordinates and obtain the restrictions on the metric; at the end of that section we shall have obtained necessary conditions on the fields in our theory for them to give non-trivial solutions to the KSEs. In section 5 we will show that the conditions obtained thus far are not only necessary but also sufficient to guarantee preserved supersymmetry. In section 6 we shall then discuss the equations that need to be satisfied in order to solve the SUGRA equations of motion. Finally, section 7 contains our conclusions.

We could not resist the temptation to include some appendices which explain the meaning and properties of the mathematical objects which are used in the gauging of $N = 2$, $d = 4$ theories: in app. A we give relations for the Pauli matrices and how to decompose the various spinorial bilinears using them. App. B deals with the gauging of isometries in Special Geometry and app. C does the same but for the hypermultiplets.

1 General gauged $N = 2$, $d = 4$ supergravity

In this section we are going to give a brief description of $N = 2$, $d = 4$ supergravity coupled to n vector supermultiplets and m hypermultiplets with gaugings of some of the isometries of the scalar manifolds associated to perturbative symmetries of the whole theory[5] using as gauge fields the fundamental (electric) vectors.[6]

The gravity multiplet of the $N = 2$, $d = 4$ theory consists of the graviton $e^a{}_\mu$, a pair of gravitinos $\psi_{I\,\mu}$ ($I = 1, 2$), which we describe as Weyl spinors, and a vector field $A^0{}_\mu$ (the *graviphoton*).

Each of the n vector supermultiplets of $N = 2$, $d = 4$ supergravity, labeled by $i, j, k = 1, \cdots, n$ contains one complex scalar Z^i, a pair of gaugini $\lambda^{I\,i}$ described as Weyl spinors, and a vector field $A^i{}_\mu$. The $\bar{n} = n + 1$ vectors $A^0{}_\mu, A^i{}_\mu$ are are described collectively by an array $A^\Lambda{}_\mu$ ($\Lambda = 0, \cdots, n$). In the ungauged theory, the scalar self-coupling is described by a non-linear σ-model with Kähler metric $\mathcal{G}_{ij^*}(Z, Z^*)$; their coupling to the vector fields by means of a complex matrix $\mathcal{N}_{\Lambda\Sigma}(Z, Z^*)$. These two couplings are related by a structure called Special Kähler Geometry.[7] In the gauged theory there are additional couplings due to the scalar potential and the covariant derivatives of the scalars that depend on the holomorphic components of the Killing vectors $k_\Lambda{}^i(Z)$ generating the isometries that have been gauged, and on the momentum map $\mathcal{P}_\Lambda(Z, Z^*)$, defined in eq. (B.18). The gauging of the isometries of the Special Kähler Geometry are described in detail in Appendix B.

Each hypermultiplet consists of 4 real scalars q (called *hyperscalars*) and 2 Weyl spinors ζ called *hyperini*. The $4m$ hyperscalars are collectively denoted by q^u ($u = 1, \ldots, 4m$) and the $2m$ hyperini

[4] As was mentioned above, the implications of the Fierz identities do not depend on the matter couplings and we shall take them as given and refer the reader to ref. [32] for more information.

[5] See *e.g.* ref. [33], the review [34] and the original works [35, 36] for more information on $N = 2$, $d = 4$ supergravities. Our conventions are explained in refs. [32, 10, 12, 14].

[6] The embedding-tensor formalism introduced in refs. [37] offers more general possibilities. Some of them have been explored in refs. [38].

[7] See *e.g.* ref. [33] or the appendix of ref. [10].

by ζ_α ($\alpha = 1, \ldots, 2m$). The $4m$ hyperscalars parametrize a Quaternionic Kähler manifold with metric $\mathsf{H}_{uv}(q)$. In the ungauged theory, the hyperscalars do not couple directly to any of the fields of the vector multiplets and their only self-coupling is determined by $\mathsf{H}_{uv}(q)$. In the gauged theory, however, there are direct couplings between the hyperscalars and the vectors and complex scalars in the scalar potential and in the covariant derivatives of the hyperscalars. These couplings depend on the the Killing vectors $\mathsf{k}_\Lambda{}^u(q)$ of the isometries that have been gauged and on the triholomorphic momentum map $\mathsf{P}_\Lambda{}^x(q)$, defined in eq. (C.12). The gauging of the isometries of the Quaternionic Kähler manifold are described in Appendix C.

The action of the bosonic fields of the theory is [33]

$$\begin{aligned} S &= \int d^4x\sqrt{|g|}\,\big[R + 2\mathcal{G}_{ij^*}\mathfrak{D}_\mu Z^i \mathfrak{D}^\mu Z^{*\,j^*} + 2\mathsf{H}_{uv}\mathfrak{D}_\mu q^u \mathfrak{D}^\mu q^v + 2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} \\ &\qquad -2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu} - V(Z,Z^*,q)\big]\ , \end{aligned} \tag{1.1}$$

where the covariant derivatives acting on the scalars are defined in eqs. (B.4) and (C.3), the vector field strengths in eq. (B.11); the scalar potential $V(Z,Z^*,q)$ is given by

$$\begin{aligned} V(Z,Z^*,q) &= g^2\left[-\tfrac{1}{4}(\Im\mathrm{m}\mathcal{N})^{-1|\Lambda\Sigma}\mathcal{P}_\Lambda\mathcal{P}_\Sigma + \tfrac{1}{2}\mathcal{L}^{*\,\Lambda}\mathcal{L}^{\Sigma}(4\mathsf{H}_{uv}\mathsf{k}_\Lambda{}^u\mathsf{k}_\Sigma{}^v - 3\mathsf{P}_\Lambda{}^x\mathsf{P}_\Sigma{}^x)\right. \\ &\qquad \left. +\tfrac{1}{2}\mathcal{G}^{ij^*}f^{\Lambda}{}_i f^{*\,\Sigma}{}_{j^*}\mathsf{P}_\Lambda{}^x\mathsf{P}_\Sigma{}^x\ \right]\ . \end{aligned} \tag{1.2}$$

The supersymmetry transformation rules of the fermions for vanishing fermions are

$$\delta_\epsilon\psi_{I\,\mu} = \mathfrak{D}_\mu\epsilon_I + \left[T^{+}{}_{\mu\nu}\varepsilon_{IJ} - \tfrac{1}{2}S^x\eta_{\mu\nu}\varepsilon_{IK}(\sigma^x)^K{}_J\right]\gamma^\nu\epsilon^J\,, \tag{1.3}$$

$$\delta_\epsilon\lambda^{Ii} = i\,\not{\mathfrak{D}}Z^i\epsilon^I + \left[\left(\not{G}^{i\,+} + W^i\right)\varepsilon^{IJ} + \tfrac{i}{2}W^{i\,x}\,(\sigma^x)^I{}_K\varepsilon^{KJ}\right]\epsilon_J\,, \tag{1.4}$$

$$\delta_\epsilon\zeta_\alpha = i\mathsf{U}_{\alpha I\,u}\,\not{\mathfrak{D}}q^u\epsilon^I + N_\alpha{}^I\epsilon_I\,, \tag{1.5}$$

where the covariant derivative acting on the spinors is given in eq. (C.35), our conventions for the Pauli matrices are described in Appendix A and where $T_{\mu\nu}$ and $G^i{}_{\mu\nu}$ are, respectively, the graviphoton and matter vector field strengths; they are defined by

$$T_{\mu\nu} \equiv 2i\mathcal{L}^\Sigma\Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}F^\Lambda{}_{\mu\nu}\,, \tag{1.6}$$

$$G^{i\,+}{}_{\mu\nu} \equiv -\mathcal{G}^{ij^*}f^{*\,\Sigma}{}_{j^*}\Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}F^\Lambda{}_{\mu\nu}\,. \tag{1.7}$$

Th so-called *fermion shifts* $S^x, W^i, W^{i\,x}, N_\alpha{}^I$ are given by

$$S^x = \tfrac{1}{2}g\mathcal{L}^\Lambda\mathsf{P}_\Lambda{}^x\,, \tag{1.8}$$

$$W^i = \tfrac{1}{2}g\mathcal{L}^{*\,\Lambda}k_\Lambda{}^i = -\tfrac{i}{2}g\mathcal{G}^{ij^*}f^{*\Lambda}{}_{j^*}\mathcal{P}_\Lambda\,, \tag{1.9}$$

$$W^{i\,x} = g\mathcal{G}^{ij^*}f^{*\,\Lambda}{}_{j^*}\mathsf{P}_\Lambda{}^x\,, \tag{1.10}$$

$$N_\alpha{}^I = g\mathsf{U}_\alpha{}^I{}_u\mathcal{L}^{*\,\Lambda}\mathsf{k}_\Lambda{}^u\,. \tag{1.11}$$

The supersymmetry transformations of the bosonic fields, also for vanishing fermions, are

$$\begin{aligned}
\delta_\epsilon e^a{}_\mu &= -\tfrac{i}{4}\bar{\psi}_{I\,\mu}\gamma^a\epsilon^I + \text{c.c.}\,, && (1.12)\\
\delta_\epsilon A^\Lambda{}_\mu &= \tfrac{1}{4}\mathcal{L}^{\Lambda\,*}\varepsilon^{IJ}\bar{\psi}_{I\,\mu}\epsilon_J + \tfrac{i}{8}f^\Lambda{}_i\varepsilon_{IJ}\bar{\lambda}^{Ii}\gamma_\mu\epsilon^J + \text{c.c.}\,, && (1.13)\\
\delta_\epsilon Z^i &= \tfrac{1}{4}\bar{\lambda}^{Ii}\epsilon_I\,, && (1.14)\\
\delta_\epsilon q^u &= \tfrac{1}{4}\mathsf{U}_{\alpha I}{}^u\bar{\zeta}^\alpha\epsilon^I + \text{c.c.}\,, && (1.15)
\end{aligned}$$

and do not depend on the gauge coupling constant g. Actually, they take the same form in the gauged and ungauged cases, a fact that will be exploited in the derivation of the KSIs.

For convenience, we denote the bosonic equations of motion by

$$\mathcal{E}_a{}^\mu \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\,,\quad \mathcal{E}_i \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta Z^i}\,,\quad \mathcal{E}_\Lambda{}^\mu \equiv \frac{1}{8\sqrt{|g|}}\frac{\delta S}{\delta A^\Lambda{}_\mu}\,,\quad \mathcal{E}^u \equiv -\frac{1}{4\sqrt{|g|}}\mathsf{H}^{uv}\frac{\delta S}{\delta q^v}\,. \tag{1.16}$$

and the Bianchi identities for the vector field strengths by

$$\mathcal{B}^{\Lambda\,\mu} \equiv \mathfrak{D}_\nu \star F^{\Lambda\,\nu\mu}\,. \tag{1.17}$$

Using the action eq. (1.1), we can calculate them to be of the form

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + 8\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,+}{}_\mu{}^\rho F^{\Sigma\,-}{}_{\nu\rho} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_{(\mu}Z^i\mathfrak{D}_{\nu)}Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i\mathfrak{D}^\rho Z^{*\,j^*}]\\
&\quad +2\mathsf{H}_{uv}\,[\mathfrak{D}_\mu q^u\mathfrak{D}_\nu q^v - \tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho q^u\mathfrak{D}_\rho q^v] + \tfrac{1}{2}g_{\mu\nu}V(Z,Z^*,q)\,, && (1.18)\\
\mathcal{E}_\Lambda{}^\mu &= \mathfrak{D}_\nu\star F_\Lambda{}^{\nu\mu} + \tfrac{1}{4}g(k_{\Lambda\,i^*}\mathfrak{D}^\mu Z^{*i^*} + k^*_{\Lambda\,i}\mathfrak{D}^\mu Z^i) + \tfrac{1}{2}g\mathsf{k}_{\Lambda\,u}\mathfrak{D}^\mu q^u\,, && (1.19)\\
\mathcal{E}^i &= \mathfrak{D}^2 Z^i + \partial^i F_\Lambda{}^{\mu\nu}\star F^\Lambda{}_{\mu\nu} + \tfrac{1}{2}\partial^i V(Z,Z^*,q)\,. && (1.20)\\
\mathcal{E}^u &= \mathfrak{D}^2 q^u + \tfrac{1}{4}\partial^u V(Z,Z^*,q)\,, && (1.21)
\end{aligned}$$

where we have defined the dual field strengths

$$F_{\Lambda\,\mu\nu} \equiv -\frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta\star F^\Lambda{}_{\mu\nu}} = 2\Re\mathrm{e}\,(\mathcal{N}^*_{\Lambda\Sigma}F^{\Sigma\,+}{}_{\mu\nu}) = \Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^\Sigma{}_{\mu\nu} + \Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}\star F^\Sigma{}_{\mu\nu}\,. \tag{1.22}$$

Combining the fundamental vector field strengths F^Λ with their magnetic duals F_Λ into a symplectic vector $\mathcal{F}^T = \big(F^\Lambda, F_\Lambda\big)$, one can rewrite many objects in a manifestly symplectic-invariant form. For instance, the graviphoton and matter field strengths are given by

$$T^+{}_{\mu\nu} = \langle\,\mathcal{V}\mid\mathcal{F}_{\mu\nu}\,\rangle \qquad\text{and}\qquad G^{i\,+}{}_{\mu\nu} = \tfrac{i}{2}\mathcal{G}^{ij^*}\langle\,\mathcal{D}_{j^*}\mathcal{V}^*\mid\mathcal{F}_{\mu\nu}\,\rangle\,, \tag{1.23}$$

where the symplectic notation is for example explained in ref. [33].

2 Supersymmetric configurations and Killing Spinor Identities

Our first goal is to find all the bosonic field configurations

$$\left\{g_{\mu\nu}(x), F^{\Lambda}{}_{\mu\nu}(x), A^{\Lambda}{}_{\mu}(x), Z^{i}(x), q^{u}(x)\right\}\ ,$$

for which the Killing Spinor Equations (KSEs) of these theories, *i.e.*

$$\delta_\epsilon \psi_{I\,\mu} \;=\; \mathfrak{D}_\mu \epsilon_I + \left[T^{+}{}_{\mu\nu}\varepsilon_{IJ} - \tfrac{1}{2}S^{x}\eta_{\mu\nu}\varepsilon_{IK}(\sigma^{x})^{K}{}_{J}\right]\gamma^{\nu}\epsilon^{J} = 0\,, \tag{2.1}$$

$$\delta_\epsilon \lambda^{Ii} \;=\; i\,\not{\mathfrak{D}}Z^{i}\epsilon^{I} + \left[\left(\not{G}^{i\,+} + W^{i}\right)\varepsilon^{IJ} + \tfrac{i}{2}W^{i\,x}\,(\sigma^{x})^{I}{}_{K}\varepsilon^{KJ}\right]\epsilon_J = 0\,, \tag{2.2}$$

$$\delta_\epsilon \zeta_\alpha \;=\; i\mathsf{U}_{\alpha I\,u}\,\not{\mathfrak{D}}q^{u}\epsilon^{I} + N_{\alpha}{}^{I}\epsilon_I = 0\,, \tag{2.3}$$

admit at least one solution ϵ_I, which is then called a *Killing spinor*. As usual in this kind of analysis, we will not assume that the Bianchi identities are satisfied by the field strengths of a configuration which should be regarded as "black boxes". Imposing the Bianchi identities will be equivalent to imposing that those black boxes are related to the potentials (which are used explicitly in gauged theories) by eq. (B.11); for the moment we will treat the vectors $A^{\Lambda}{}_{\mu}$ and the vector field strengths $F^{\Lambda}{}_{\mu\nu}$ as independent fields. We will impose the Bianchi identities together with the equations of motion after we have found the supersymmetric configurations and at the end we will have supersymmetric solutions determined by the independent fields $\left\{g_{\mu\nu}(x), A^{\Lambda}{}_{\mu}(x), Z^{i}(x), q^{u}(x)\right\}$.

We start by studying the integrability conditions of the above KSEs: using the supersymmetry transformation rules of the bosonic fields eqs. (1.12–1.14) and using the results of refs. [7, 6] we can derive the following KSIs satisfied by any field configuration admitting Killing spinors:

$$\mathcal{E}_{a}{}^{\mu}\gamma^{a}\epsilon^{I} - 4i\varepsilon^{IJ}\mathcal{L}^{\Lambda}\mathcal{E}_{\Lambda}{}^{\mu}\epsilon_J \;=\; 0\,, \tag{2.4}$$

$$\mathcal{E}^{i}\epsilon^{I} - 2i\varepsilon^{IJ}f^{*\,i\Lambda}\,\not{\mathcal{E}}_{\Lambda}\epsilon_J \;=\; 0\,, \tag{2.5}$$

$$\mathcal{E}^{u}\,\mathsf{U}^{\alpha I}{}_{u}\epsilon_I \;=\; 0\,, \tag{2.6}$$

The vector field Bianchi identities eq. (1.17) do not appear in these relations because the procedure used to derive them assumes the existence of the vector potentials, and therefore uses the vanishing of the Bianchi identities.

It is convenient to treat the Maxwell equations and Bianchi identities on an equal footing as to preserve the symplectic covariance of the theory, which means having a formally electric-magnetic duality-covariant version of the above KSIs. This version can be found by performing duality rotations on the above identities or from the integrability conditions of the KSEs (see *e.g.* [10]). Both procedures give exactly the same symplectic-invariant result, namely

$$\mathcal{E}_{a}{}^{\mu}\gamma^{a}\epsilon_I - 4i\langle\,\mathcal{E}^{\mu}\mid\mathcal{V}\,\rangle\varepsilon_{IJ}\epsilon^{J} \;=\; 0\,, \tag{2.7}$$

$$\mathcal{E}^{i}\epsilon^{I} + 2i\langle\,\not{\mathcal{E}}\mid\mathcal{U}^{*\,i}\,\rangle\varepsilon^{IJ}\epsilon_J \;=\; 0\,, \tag{2.8}$$

$$\mathcal{E}^{u}\,\mathsf{U}^{\alpha I}{}_{u}\epsilon_I \;=\; 0\,, \tag{2.9}$$

where $\mathcal{E}^\mu$ is a symplectic vector containing the Maxwell equations and Bianchi identities

$$\mathcal{E}^\mu \equiv \left(\begin{array}{c} \mathcal{B}^{\Lambda\,\mu} \\ \mathcal{E}_\Lambda{}^\mu \end{array} \right) . \tag{2.10}$$

Acting on these identities with gamma matrices and conjugate spinors from the left, we get identities involving the equations of motion and tensors which are the bilinears of the Killing spinors. As mentioned in the introduction, there are two cases to be considered, the sexer being the causal nature of the vector bilinear $V^a \equiv i\bar{\epsilon}^I\gamma^a\epsilon_I$, namely the timelike and the null case. In the timelike case (the only one we are going to consider in this paper) we can use $V^a/|V|$ as the component e^0 of an orthonormal frame, obtaining the identities

$$\begin{aligned}
\mathcal{E}^{0m} &= \mathcal{E}^{mn} = 0\,, && (2.11)\\
\langle\, \mathcal{V}/X \mid \mathcal{E}^0 \,\rangle &= \tfrac{1}{4}|X|^{-1}\mathcal{E}^{00}\,, && (2.12)\\
\langle\, \mathcal{V}/X \mid \mathcal{E}^m \,\rangle &= 0\,, && (2.13)\\
\langle\, \mathcal{U}^*_{i^*} \mid \mathcal{E}^0 \,\rangle &= \tfrac{1}{2}e^{-i\alpha}\mathcal{E}_{i^*}\,, && (2.14)\\
\langle\, \mathcal{U}^*_{i^*} \mid \mathcal{E}^m \,\rangle &= 0\,, && (2.15)\\
\mathcal{E}^u &= 0\,, && (2.16)
\end{aligned}$$

where $X \equiv \frac{1}{2}\varepsilon^{IJ}\bar{\epsilon}_I\epsilon_J$ is the scalar bilinear and α is its phase [32]. These identities imply that [4, 6]

1. All the supersymmetric configurations automatically satisfy all the equations of motion except for $\mathcal{E}^0$ and $\mathcal{E}^{00}$ and also the Bianchi identities.
2. We will only need to impose $\mathcal{E}^0 = 0$ on the supersymmetric configurations in order to have a solution of all the classical equations of motion and Bianchi identities.

3 Killing equations for the bilinears

From the gravitino supersymmetry transformation rule eq. (1.3), using the decompositions eqs. (A.9)-(A.11) we get the independent equations

$$\mathfrak{D}_\mu X = iV^\nu T^+{}_{\nu\mu} + \tfrac{i}{\sqrt{2}} S^x V^x{}_\mu\,, \tag{3.1}$$

$$\nabla_{(\mu} V_{\nu)} = 0\,, \tag{3.2}$$

$$d\hat{V} = 4iX\hat{T}^{*-} - \sqrt{2} S^{*x}\hat{\Phi}^x + \text{c.c.}\,, \tag{3.3}$$

$$\mathfrak{D}_{(\mu} V^x{}_{\nu)} = T^{*-}{}_{(\mu|\rho}\Phi^x{}_{|\nu)}{}^\rho + \tfrac{i}{\sqrt{2}} X S^{*x} g_{\mu\nu} + \text{c.c.}\,, \tag{3.4}$$

$$\mathfrak{D}\hat{V}^x = -i\epsilon^{xyz} S^{*y}\hat{\Phi}^z + \text{c.c.}\,, \tag{3.5}$$

where we denote differential forms with hats, and the $\mathrm{SU}(2)$-covariant derivative is

$$\mathfrak{D}\hat{V}^x = d\hat{V}^x + \epsilon^{xyz}\hat{\mathsf{A}}^y \wedge \hat{V}^z\,. \tag{3.6}$$

Eq. (3.2) indicates that V is, as usual in SUGRA, a timelike Killing vector. According to eq. (3.4), the vectors V^x are not, in general. However, for vanishing graviphoton field strength, they are conformal Killing vectors. The equations for $d\hat{V}$ (3.3) and $\mathfrak{D}\hat{V}^x$ (3.5) will be used and analyzed later on.

From the gauginos' supersymmetry transformation rules, eqs. (1.4), we get

$$0 = V^I{}_K{}^\mu \mathfrak{D}_\mu Z^i + \varepsilon^{IJ}\Phi_{KJ}{}^{\mu\nu} G^{i\,+}{}_{\mu\nu} + W^i\delta^I{}_K + XW^{i\,I}{}_K\,, \tag{3.7}$$

$$\begin{aligned} 0 &= iX^*\varepsilon^{KI}\mathfrak{D}^\mu Z^i + i\Phi^{KI\,\mu\nu}\mathfrak{D}_\nu Z^i - 4i\varepsilon^{IJ} G^{i\,+\,\mu}{}_\nu V^K{}_J{}^\nu \\ &\quad -iW^i\varepsilon^{IJ}V^K{}_J{}^\mu - iW^{i\,IJ}V^K{}_{J\,\mu}\,. \end{aligned} \tag{3.8}$$

The trace of the first equation gives

$$V^\mu \mathfrak{D}_\mu Z^i + 2XW^i = 0\,, \tag{3.9}$$

while the antisymmetric part of the second equation gives

$$2X^*\mathfrak{D}_\mu Z^i + 4G^{i\,+}{}_{\mu\nu}V^\nu + W^i V_\mu - W^{i\;J}{}_K V^K{}_{J\,\mu} = 0\,. \tag{3.10}$$

From eqs. (3.1) and (3.10) we get

$$V^\nu T^+{}_{\nu\mu} = -i\mathfrak{D}_\mu X - \tfrac{1}{\sqrt{2}} S^x V^x{}_\mu\,, \tag{3.11}$$

$$V^\nu G^{i\,+}{}_{\nu\mu} = \tfrac{1}{2}X^*\mathfrak{D}_\mu Z^i + \tfrac{1}{4}W^i V_\mu - \tfrac{i}{4\sqrt{2}} W^{ix} V^x{}_\mu\,. \tag{3.12}$$

The consistency of these expressions requires

$$V^\mu \mathfrak{D}_\mu X = 0\,, \tag{3.13}$$

and eq. (3.9), respectively. Upon using the Special Geometry completeness relation [33]

$$F^{\Lambda\,+} = i\mathcal{L}^{*\,\Lambda}T^{+} + 2f^{\Lambda}{}_{i}G^{i\,+}\,, \tag{3.14}$$

we obtain from eqs. (3.11) and (3.12), first of all

$$V^{\nu}F^{\Lambda\,+}{}_{\nu\mu} = \mathcal{L}^{*\,\Lambda}\mathfrak{D}_{\mu}X + X^{*}\mathfrak{D}_{\mu}\mathcal{L}^{\Lambda} + \tfrac{i}{8}g\Im\mathrm{m}\mathcal{N}^{-1|\Lambda\Sigma}(\mathcal{P}_{\Sigma}V_{\mu} + \sqrt{2}\,\mathsf{P}_{\Sigma}{}^{x}V^{x}{}_{\mu})\,. \tag{3.15}$$

Then, using $F_{\Lambda}{}^{+} = \mathcal{N}^{*}_{\Lambda\Sigma}F^{\Sigma\,+}$, we get for the symplectic vector of field strengths $\mathcal{F}^{T} \equiv \left(\, F^{\Lambda}, F_{\Lambda}\right)$

$$V^{\nu}\mathcal{F}^{+}{}_{\nu\mu} = \mathcal{V}^{*}\mathfrak{D}_{\mu}X + X^{*}\mathfrak{D}_{\mu}\mathcal{V} \;-\; \tfrac{i}{8}g\,\Omega^{-1}\left(\mathcal{M} + i\Omega\right)\,\left[\mathcal{P}V_{\mu} + \sqrt{2}\mathsf{P}^{x}V^{x}{}_{\mu}\right]\,, \tag{3.16}$$

where $\mathcal{M}$ and Ω are the symplectic matrices

$$\mathcal{M} \equiv \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ & \\ -I^{-1}R & I^{-1} \end{pmatrix}\,, \qquad \Omega \equiv \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}\,, \tag{3.17}$$

and we have defined

$$I_{\Lambda\Sigma} \equiv \Im\mathrm{m}\left(\mathcal{N}_{\Lambda\Sigma}\right)\,, \qquad R_{\Lambda\Sigma} \equiv \Re\mathrm{e}\left(\mathcal{N}_{\Lambda\Sigma}\right)\,, \qquad I^{\Lambda\Sigma}I_{\Sigma\Omega} = \delta^{\Sigma}{}_{\Omega}\,; \tag{3.18}$$

furthermore, we have introduced the following symplectic vectors for the momentum maps

$$\mathcal{P} \;=\; \begin{pmatrix} 0 \\ \mathcal{P}_{\Lambda} \end{pmatrix}\,, \qquad \mathsf{P}^{x} \;=\; \begin{pmatrix} 0 \\ \mathsf{P}_{\Lambda}{}^{x} \end{pmatrix}\,. \tag{3.19}$$

Had we used the embedding-tensor formalism [37], none of the components of these symplectic vectors would have vanished and we would have obtained manifestly symplectic-invariant expressions; using only the fundamental (electric) 1-forms as gauge fields, however, kills off half of the components, as we have seen above.

After some straightforward manipulations we obtain the general form of the electric and magnetic field strengths

$$\begin{aligned} \mathcal{F} \;\;=\;\; & -\tfrac{1}{2}\,\mathfrak{D}[\mathcal{R}\,\hat{V}] \;+\; \frac{g}{8\sqrt{2}|X|^{2}}\,\mathsf{P}^{x}\,\hat{V}\wedge\hat{V}^{x} \\ & -\tfrac{1}{2}\star\left\{\hat{V}\wedge\left[\,\mathfrak{D}\mathcal{I} \;-\; \sqrt{2}g\,\left(\mathcal{R}\,\langle\mathcal{R}|\mathsf{P}^{x}\rangle \;-\; \frac{1}{8|X|^{2}}\,\Omega^{-1}\mathcal{M}\,\mathsf{P}^{x}\right)\,\hat{V}^{x}\,\right]\right\}\,, \end{aligned} \tag{3.20}$$

where following ref. [10] we have defined

$$\mathcal{V}/X \;\equiv\; \mathcal{R} \;+\; i\mathcal{I}\,. \tag{3.21}$$

Let us now consider the hyperini's KSE: it is convenient to rewrite it as

$$\not{\mathfrak{D}}q^{u}\epsilon^{I} - i\mathsf{K}^{x\,u}{}_{v}\sigma^{x}{}_{J}{}^{I}\,\not{\mathfrak{D}}q^{v}\epsilon^{J} - ig\varepsilon^{IJ}\mathcal{L}^{*\,\Lambda}\mathsf{k}_{\Lambda}{}^{u}\epsilon_{J} + \tfrac{1}{2}g\mathcal{L}^{*\,\Lambda}\mathfrak{D}^{u}\mathsf{P}_{\Lambda}{}^{x}\sigma^{x\,IJ}\epsilon_{J} = 0\,. \tag{3.22}$$

We only get one independent equation for the bilinears:

$$V^I{}_K{}^\mu \mathfrak{D}_\mu q^u - i\mathsf{K}^{x\,u}{}_v \sigma^x{}_J{}^I V^J{}_K{}^\mu \mathfrak{D}_\mu q^v + gX\delta^I{}_K \mathcal{L}^{*\,\Lambda} \mathsf{k}_\Lambda{}^u + \tfrac{i}{2} gX\mathcal{L}^{*\,\Lambda} \mathfrak{D}^u \mathsf{P}_\Lambda{}^x \sigma^{x\,I}{}_K = 0\,. \quad (3.23)$$

The trace of this equation is

$$V^\mu \mathfrak{D}_\mu q^u - i\sqrt{2}\mathsf{K}^{x\,u}{}_v V^{x\,\mu} \mathfrak{D}_\mu q^v + 2gX\mathcal{L}^{*\,\Lambda} \mathsf{k}_\Lambda{}^u = 0\,, \quad (3.24)$$

and its real and imaginary parts are

$$V^\mu \mathfrak{D}_\mu q^u + 2g|X|^2 \mathcal{R}^\Lambda \mathsf{k}_\Lambda{}^u = 0\,, \quad (3.25)$$

$$\mathsf{K}^{x\,u}{}_v V^{x\,\mu} \mathfrak{D}_\mu q^v + \sqrt{2} g|X|^2 \mathcal{I}^\Lambda \mathsf{k}_\Lambda{}^u = 0\,. \quad (3.26)$$

The rest of the equations that can be obtained from eq. (3.23) can also be obtained from these two. In particular, we can get from eq. (3.26)

$$V^{x\,\mu} \mathfrak{D}_\mu q^u + \varepsilon_{xyz} \mathsf{K}^{y\,u}{}_v V^{z\,\mu} \mathfrak{D}_\mu q^v + \tfrac{1}{\sqrt{2}} g|X|^2 \mathcal{I}^\Lambda \mathfrak{D}^u \mathsf{P}_\Lambda{}^x = 0\,. \quad (3.27)$$

In order to make further progress we must introduce coordinates and obtain information about the metric.

4 The metric

We define a time coordinate t associated to the timelike Killing vector V by

$$V^\mu \partial_\mu \equiv \sqrt{2}\partial_t\,. \quad (4.1)$$

Then, by choosing the gauge fixing condition

$$V^\mu A^\Lambda{}_\mu = \sqrt{2} A^\Lambda{}_t = -2|X|^2 \mathcal{R}^\Lambda\,, \quad (4.2)$$

we can solve eqs. (3.9,3.13) and (3.25) by taking all the scalar fields and the function X to be time-independent,[8] *i.e.*

$$\partial_t Z^i = \partial_t X = \partial_t q^u = 0\,. \quad (4.6)$$

The definition eq. (4.1) and the Fierz identity $V^2 = 4|X|^2$ imply that $\hat{V}$ must take the form

[8]The consistency of this gauge choice in all the equations derived from the KSEs requires the use of several identities that can be derived from the generic expression of the momentum map $\mathcal{P}_\Lambda$ eq. (B.51), which is equivalent to

$$\mathcal{P}_\Lambda = 2|X|^2 f_{\Lambda\Sigma}{}^\Omega \left(\mathcal{R}^\Sigma \mathcal{R}_\Omega + \mathcal{I}^\Sigma \mathcal{I}_\Omega\right)\,, \quad (4.3)$$

the property eq. (B.45) and $\mathfrak{D}\mathcal{M}_\Lambda = \mathcal{N}^*_{\Lambda\Sigma}\mathfrak{D}\mathcal{L}^\Sigma$. These properties are

$$f_{\Lambda\Sigma}{}^\Omega \mathcal{R}^\Sigma \mathcal{R}_\Omega = f_{\Lambda\Sigma}{}^\Omega \mathcal{I}^\Sigma \mathcal{I}_\Omega = I_{\Lambda\Gamma} f_{\Sigma\Omega}{}^\Gamma \mathcal{R}^\Sigma \mathcal{I}^\Omega\,, \quad (4.4)$$

$$f_{\Lambda\Sigma}{}^\Omega \mathcal{R}^\Sigma \mathcal{I}_\Omega = -f_{\Lambda\Sigma}{}^\Omega \mathcal{I}^\Sigma \mathcal{R}_\Omega = -R_{\Lambda\Gamma} T^{\Gamma\Delta} f_{\Delta\Sigma}{}^\Omega \mathcal{R}^\Sigma \mathcal{R}_\Omega\,. \quad (4.5)$$

$$\hat{V} = 2\sqrt{2}|X|^2(dt + \hat{\omega})\,, \tag{4.7}$$

where $\hat{\omega}$ is a spatial 1-form, which by definition, must satisfy

$$d\hat{\omega} = \tfrac{1}{2\sqrt{2}} d(|X|^{-2}\hat{V})\,. \tag{4.8}$$

eqs. (3.3,3.1) and some straightforward manipulations imply that $\hat{\omega}$ must satisfy

$$d\hat{\omega} = -\tfrac{i}{2\sqrt{2}} \star \left[\left(X\mathfrak{D}X^* - X^*\mathfrak{D}X \ + \ ig\sqrt{2}|X|^2 \mathcal{R}^\Lambda \mathsf{P}_\Lambda{}^x \hat{V}^x \right) \wedge \frac{\hat{V}}{|X|^4} \right] . \tag{4.9}$$

Since the $\hat{V}^x$s are not exact, we cannot simply define coordinates by putting $\hat{V}^x \equiv dx^x$. We can, however, still use them to construct the metric: using

$$g_{\mu\nu} = 2V^{-2}[V_\mu V_\nu - V^J{}_{I\,\mu} V^I{}_{J\,\nu}]\,, \tag{4.10}$$

and the decomposition eq. (A.9), we find that the metric can be written in the form

$$ds^2 = \frac{1}{4|X|^2} \hat{V} \otimes \hat{V} - \frac{1}{2|X|^2} \delta_{xy} \hat{V}^x \otimes \hat{V}^y\,. \tag{4.11}$$

The $\hat{V}^x$ are mutually orthogonal and also orthogonal to $\hat{V}$, which means that they can be used as a Dreibein for a 3-dimensional Euclidean metric

$$\delta_{xy} \hat{V}^x \otimes \hat{V}^y \ \equiv \ \gamma_{\underline{mn}} dx^m dx^n\,, \tag{4.12}$$

where we introduced the remaining 3 spatial coordinates x^m ($m = 1, 2, 3$). The 4-dimensional metric takes the coordinate-form

$$ds^2 \ = \ 2|X|^2(dt + \hat{\omega})^2 - \frac{1}{2|X|^2} \gamma_{\underline{mn}} dx^m dx^n\,. \tag{4.13}$$

In what follows we will use the Vierbein basis

$$e^0 = \frac{1}{2|X|} \hat{V}\,, \qquad e^x = \frac{1}{\sqrt{2}|X|} \hat{V}^x\,, \tag{4.14}$$

that is

$$(e^a{}_\mu) = \begin{pmatrix} \sqrt{2}|X| & \sqrt{2}|X|\omega_{\underline{m}} \\ 0 & \frac{1}{\sqrt{2}|X|} V^x{}_{\underline{m}} \end{pmatrix}, \qquad (e^\mu{}_a) = \begin{pmatrix} \frac{1}{\sqrt{2}|X|} & -\sqrt{2}|X|\omega_x \\ 0 & \sqrt{2}|X| V_x{}^{\underline{m}} \end{pmatrix}. \tag{4.15}$$

where $V_x{}^{\underline{m}}$ is the inverse Dreibein $V_x{}^{\underline{m}} V^y{}_{\underline{m}} = \delta^y{}_x$ and $\omega_x = V_x{}^{\underline{m}} \omega_{\underline{m}}$. Observe that we can raise and lower flat 3-dimensional indices with δ_{xy} and δ^{xy}, whence their position is rather irrelevant. We shall also adopt the convention that, from now on, all objects with flat or curved 3-dimensional indices refer to the above Dreibein and the corresponding metric.

Using these conventions, we see that eq. (4.9) takes the 3-dimensional form

$$(d\hat{\omega})_{xy} = -\frac{1}{2|X|^2} \varepsilon_{xyz} \left\{ i \left(\frac{\tilde{\mathfrak{D}}_z X}{X} - \frac{\tilde{\mathfrak{D}}_z X^*}{X^*} \right) - \sqrt{2} g \left\langle \mathcal{R} \mid \mathsf{P}^z \right\rangle \right\}, \tag{4.16}$$

or using the symplectic vectors defined in eq. (3.21)

$$(d\hat{\omega})_{xy} \;=\; 2\varepsilon_{xyz}\left\{\langle\mathcal{I}\mid\tilde{\mathfrak{D}}_z\mathcal{I}\rangle \;+\; \frac{g}{2\sqrt{2}|X|^2}\langle\,\mathcal{R}\mid \mathsf{P}^z\,\rangle\right\}\,, \tag{4.17}$$

where $\tilde{\mathfrak{D}}$ is the covariant derivative w.r.t. the effective 3-dimensional gauge connection

$$\tilde{A}^{\Lambda}{}_{\underline{m}} \;\equiv\; A^{\Lambda}{}_{\underline{m}} - \omega_{\underline{m}}A^{\Lambda}{}_t \;=\; A^{\Lambda}{}_{\underline{m}} + \sqrt{2}|X|^2\mathcal{R}^{\Lambda}\omega_{\underline{m}}\,. \tag{4.18}$$

Let us now consider eq. (3.5): the mixed indices part takes on the form, using the gauge fixing eq. (4.2),

$$\partial_t V^{x}{}_{\underline{m}} = 0\,, \tag{4.19}$$

while the purely spatial part takes the form

$$d\hat{V}^x + \epsilon^{xyz}\tilde{\hat{A}}^y\wedge\hat{V}^z + \hat{T}^x = 0\,, \tag{4.20}$$

where

$$\tilde{\hat{A}}^{x}{}_{\underline{m}} \;\equiv\; \mathsf{A}^{x}{}_{\underline{m}} - \sqrt{2}g|X|^2\,\langle\,\tilde{A}_{\underline{m}}\mid \mathsf{P}^x\,\rangle\omega_{\underline{m}} = \mathsf{A}^{x}{}_{\underline{m}} - g\,\langle\,\tilde{A}_{\underline{m}}\mid \mathsf{P}^x\,\rangle\,, \tag{4.21}$$

$$\hat{T}^x \;=\; -\tfrac{1}{2\sqrt{2}}g\epsilon^{xyz}\langle\,\mathcal{I}\mid \mathsf{P}^y\,\rangle\,\epsilon^{zvw}\hat{V}^v\wedge\hat{V}^w\,. \tag{4.22}$$

The above equation can be interpreted as Cartan's first structure equations for the Dreibein $\hat{V}^x$, the $\mathrm{SU}(2)$-connection 1-form $\tilde{\hat{A}}^x$ and the torsion 2-form $\hat{T}^x$. It can be solved for the spin connection as a function of the Dreibein and torsion, *i.e.*

$$\varpi_{xyz}(V) = -\varepsilon_{yzw}\tilde{A}^{w}{}_x - K_{xyz}(T)\,, \tag{4.23}$$

where $\varpi_{xyz}(V)$ is the standard 3-dimensional Levi-Cività connection 1-form (which is completely determined by the Dreibein), and $K_{xyz}(T)$ is the contorsion 1-form, to wit

$$K_{xyz} = \tfrac{1}{2}\{T_{xzy} + T_{yzx} - T_{xyz}\} = -\sqrt{2}g\,\langle\,\mathcal{I}\mid \mathsf{P}^{[y}\,\rangle\,\delta^{z]x}\,. \tag{4.24}$$

This condition relates the spin connection of the 3-dimensional space with the pullback of the $\mathrm{SU}(2)$-connection, the gauge connection and the complex scalars. In the ungauged case, considered in ref. [12], this complicated relation reduces to a straightforward relation between the first two.

Let us summarize our results: we have shown that

1. The metric of a bosonic field configuration of $N=2$, $d=4$ supergravity $A^{\Lambda}{}_{\mu}, Z^i, q^u$ can be written in the conformastationary form eq. (4.13) where the spatial 1-form $\hat{\omega}$ satisfies eq. (4.17) and the spin connection of spatial 3-dimensional metric $\gamma_{\underline{mn}}$ is related to the pullback of the quaternionic-Kähler $\mathrm{SU}(2)$ connection $\mathsf{A}^{I}{}_{J\,\mu}$ and the gauge connection by eqs. (4.21,4.18) and (4.23).

2. The vector field strengths must take the form that can be derived from eq. (3.15).

3. The covariant derivatives of the hyperscalars must satisfy eqs. (3.25) and (3.26). In the gauge eq. (4.2), eq. (3.25) just states that the hyperscalars are time-independent.

4. The complex scalars Z^i must satisfy eq. (3.9) and in the gauge eq. (4.2) they are also time-independent. Observe that there are no further equations for them.

In the next section we are going to show that the necessary conditions that we have just found are also sufficient to have unbroken supersymmetry.

5 Killing spinor equations: necessary is also sufficient

Let us consider first the gaugini KSE eq. (2.2): by straightforwardly expanding and manipulating the ingredients one can put it in the following form

$$\delta_\epsilon \lambda^{Ii} \;=\; i\sqrt{2}|X|\gamma^x \tilde{\mathfrak{D}}_x Z^i \;\left(\Pi^0\epsilon\right)^I \;-\; ie^{i\alpha}W^i\gamma^0\;\left(\Pi^0\epsilon\right)^I \;-\; iW^{ix}\gamma^{0x}\varepsilon^{IL}\,\Pi^x{}_L{}^K\,\epsilon_K\;, \tag{5.1}$$

where we have defined, as was indicated before, $X = e^{i\alpha}\,|X|$ and

$$\left(\Pi^0\epsilon\right)^I \quad\equiv\quad \epsilon^I \;+\; ie^{-i\alpha}\gamma^0\,\varepsilon^{IJ}\epsilon_J\;, \tag{5.2}$$

$$\tag{5.3}$$

$$\Pi^{x\,I}{}_J \quad=\quad \tfrac{1}{2}\left[\delta^I{}_J \;+\; \gamma^{0(x)}(\sigma^{(x)})^I{}_J\right] \qquad\text{(no sum)}\,. \tag{5.4}$$

The gaugini KSE, then, will be solved if we impose the projections

$$\left(\Pi^0\epsilon\right)^I = 0\,, \qquad \Pi^{x\,I}{}_J\epsilon_I = 0\,, \tag{5.5}$$

for all x for which $W^{ix} \neq 0$. The crucial properties of the Π^x are

$$(\Pi^x)^2 \;=\; \Pi^x \;\;,\;\; \mathrm{Tr}\,(\Pi^x) \;=\; 4 \qquad\text{and}\qquad [\,\Pi^x\,,\,\Pi^y\,] \;=\; 0\,, \tag{5.6}$$

which guarantees that all 3 constraints $\Pi^{x\,I}{}_J\epsilon_I = 0$ can, if necessary, be consistently imposed at the same time. Furthermore, the properties of the Pauli matrices (see Appendix A)[9] ensure that these constraints are consistent with the fourth constraint, namely $\left(\Pi^0\epsilon\right)^I = 0$.

Having identified the pertinent projection operators, the remaining checks of the KSEs are straightforward and we will be brief: the analysis of the hyperini variation (2.3) implies that eq. (3.26) must be satisfied and the 0-direction of the gravitino variation (2.1) implies that the Killing spinors are time-independent. The analysis in the spacelike directions of the gravitino variation is best expressed in terms of the Kähler-weight zero spinor η_I defined by $\epsilon_I = X^{1/2}\eta_I$. The parts of said variation that do not cancel straightforwardly are

$$0 \;=\; \partial_x\eta_I \;+\; \left[\;\tfrac{1}{2}\varpi_{xzz'}\,\varepsilon^{yzz'} \;+\; \tilde{\hat{A}}^y{}_x \;-\; \tfrac{1}{\sqrt{2}}g\varepsilon^{xzy}\,\mathcal{I}^\Lambda\mathsf{P}_\Lambda{}^z\;\right]\;\tfrac{i}{2}\,(\sigma^y)^J{}_I\eta_J\;. \tag{5.7}$$

The identification of the spin connection in eqs. (4.23) and (4.24), however, implies that the second term on the right hand side vanishes, whence preserved supersymmetry implies that η_I is constant.

Summarizing and reformulating the results in this section we see that the Killing spinor takes on the form $\epsilon_I = X^{1/2}\eta_I$, where η_I is a constant spinor satisfying

$$\Pi^{x\,J}{}_I\,\eta_J \;=\; 0 \qquad\text{and}\qquad 0 \;=\; \eta^I \;+\; i\gamma^0\,\varepsilon^{IJ}\eta_J\;, \tag{5.8}$$

[9] This is easily seen to be true by making use of the identity $\varepsilon^{IL}\,\gamma^0\,\Pi^{x\,K}_L = \Pi^{xI}{}_L\,\gamma^0\,\varepsilon^{LK}$ which expresses the fact that for the Π^x complex conjugation is not the same as raising and lowering $SU(2)$ indices.

the last restriction being a rescaled version of the constraint (5.2).

Since we must generically impose 4 compatible projection operators, each of which is able to project out half of the components of the Killing spinor, and we have 8 supercharges at our disposal we naively should conclude that we must end up with a solution that has no supersymmetry whatsoever. That this is not the case is due to the structure of the Π^x's and the chirality of the Killing spinor: it is easy to see that if we impose any pair of them, say $\left(\Pi^1\epsilon\right)_I = 0$ and $\left(\Pi^2\epsilon\right)_I$, then the spinor automatically satisfies the third one, *i.e.* $\left(\Pi^3\epsilon\right)_I = 0$. This then means that the configurations that we have obtained are $\frac{1}{8}$-BPS.

6 Timelike supersymmetric solutions

The KSIs imply that the supersymmetric configurations that satisfy the zeroth components of the Maxwell equations and (the Hodge dual of) the Bianchi identities solve all the equations of motion of the theory.

The zeroth component of (the Hodge dual of) the Bianchi identities is just the Bianchi identity of the effective 3-dimensional field strength $\tilde{F}^{\Lambda}{}_{xy}$ which has the following 3-dimensional expression:

$$\tilde{F}^{\Lambda}{}_{xy} = -\tfrac{1}{\sqrt{2}}\varepsilon_{xyz}\{\tilde{\mathfrak{D}}_z\mathcal{I}^{\Lambda} + g\mathcal{B}^{\Lambda}{}_z\}\,, \tag{6.1}$$

where

$$\mathcal{B}^{\Lambda}{}_z \equiv \sqrt{2}\left[\mathcal{R}^{\Lambda}\mathcal{R}^{\Sigma} + \frac{1}{8|X|^2}I^{\Lambda\Sigma}\right]\mathsf{P}_{\Sigma}{}^{z}\,. \tag{6.2}$$

The above equation is a generalization of the well-known Bogomol'nyi equation of Yang-Mills theories to an (almost arbitrary) 3-dimensional background metric $\gamma_{\underline{mn}}$ and with an extra term. If we find $\tilde{A}^{\Lambda}{}_{\underline{m}}, \mathcal{I}^{\Lambda}, B^{\Lambda}{}_x$ solving that equation, then we have found a $\tilde{A}^{\Lambda}{}_{\underline{m}}$ that gives rise to the field strength $\tilde{F}^{\Lambda}{}_{\underline{mn}}$ with the form prescribed by supersymmetry and the 3-dimensional Bianchi identity and, therefore, the zeroth component of the 4-dimensional one, are automatically satisfied.

The integrability equation of the Bogomol'nyi equation is a generalization of the gauge-covariant Laplace equation for the $\mathcal{I}^{\Lambda}$:

$$\tilde{\mathfrak{D}}^2\mathcal{I}^{\Lambda} + g\tilde{\mathfrak{D}}_x\mathcal{B}^{\Lambda}{}_x = 0\,. \tag{6.3}$$

Observe that in the above equation the covariant derivatives not only contain the gauge part acting on the Λ-indices, but also the spin connection for the 3-dimensional base space, which is constrained by eq. (4.23)[10]. In the ungauged, Abelian cases, the $\mathcal{I}^{\Lambda}$ are just harmonic functions on $\mathbb{R}^3$.

Let us now consider the zeroth component of the Maxwell equations, which can be written as a sort of Bianchi identity for the dual field strengths F_{Λ}: a lengthy calculation shows that the equation of motion leads to

$$-\tfrac{1}{\sqrt{2}}\varepsilon_{xyz}\tilde{\mathfrak{D}}_x\tilde{F}_{\Lambda\,yz} = \tfrac{1}{2\sqrt{2}}g\varepsilon_{xyz}(d\hat{\omega})_{xy}\mathsf{P}^{z}_{\Lambda} + \tfrac{1}{2}g^2 f_{\Lambda(\Omega}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Sigma}\,\mathcal{I}^{\Omega}\mathcal{I}^{\Delta}\mathcal{I}_{\Sigma} + \frac{g^2}{4|X|^2}\mathcal{R}^{\Sigma}\mathsf{k}_{\Lambda\,u}\mathsf{k}_{\Sigma}{}^{u}\,, \tag{6.4}$$

where we have defined[11]

[10] Observe that the components of the Abelian Bianchi identity w.r.t. a curved frame reads $\nabla_{[a}F_{bc]} = 0$, the extension to a non-Abelian one being obvious.

[11] $\tilde{F}_{\Lambda\,xy}$ is strictly given by this definition because there are no dual 1-forms A_{Λ} in this formulation.

$$\tilde{F}_{\Lambda\, xy} \equiv -\tfrac{1}{\sqrt{2}}\varepsilon_{xyz}\left\{\tilde{\mathfrak{D}}_x\mathcal{I}_\Lambda + g\mathcal{B}_{\Lambda\, x}\right\} , \tag{6.5}$$

$$\mathcal{B}_{\Lambda\, x} \equiv \sqrt{2}\left[\mathcal{R}_\Lambda\mathcal{R}^\Sigma + \frac{1}{8|X|^2}\, R_{\Lambda\Gamma}I^{\Gamma\Sigma}\right]\mathsf{P}^x_\Sigma\ . \tag{6.6}$$

If we use eq. (4.17), which defines the 1-form $\hat{\omega}$ to express the equation, as much as possible, in terms of $\mathcal{R}$ and $\mathcal{I}$, we get

$$\begin{aligned} -\tfrac{1}{\sqrt{2}}\varepsilon_{xyz}\tilde{\mathfrak{D}}_x\tilde{F}_{\Lambda\, yz} &= \tfrac{1}{\sqrt{2}}g\langle\mathcal{I}\mid\tilde{\mathfrak{D}}_x\mathcal{I}\rangle\ \mathsf{P}^x_\Lambda + \tfrac{1}{2}g^2 f_{\Lambda(\Omega}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Sigma}\ \mathcal{I}^\Omega\mathcal{I}^\Delta\mathcal{I}_\Sigma \\ &\quad + \tfrac{g^2}{4|X|^2}\mathcal{R}^\Sigma\left[\mathsf{k}_{\Lambda\, u}\mathsf{k}_\Sigma{}^u - \mathsf{P}_\Lambda{}^x\mathsf{P}_\Sigma{}^x\ \right]\ . \end{aligned} \tag{6.7}$$

Observe that the above equation reduces in the hyperless case, *i.e.* $\mathsf{P}^x_\Lambda = 0$, to the expression given in [14, 23].

7 Conclusions

In this article we have obtained the form of the most general supersymmetric solution to gauged $N = 2$, $d = 4$ supergravity coupled to YM-vector multiplets and hypermultiplets and showed that we are generically dealing with $\frac{1}{8}$-BPS solutions. The generic form of the solutions is the one already known from earlier investigations, but there are some fine differences; for example in ungauged case the base space is just $\mathbb{R}^3$ and in (Abelian) gauged SUGRA the base space becomes torsionful [15], or said differently it must have a non-trivial $\mathrm{SO}(2)$ holonomy in order to be able to kill off the effective $\mathrm{U}(1)$ gauge field induced on the base space. In our case, see eq. (4.23), we have to face in general a base space with $\mathrm{SO}(3)$ holonomy as we have to kill off an effective $\mathrm{SU}(2)$ gauge field.

Clearly, the general equations that need to be solved, such as the generalized Bogomol'nyi equation in eq. (6.1), look daunting and a general solution is out of reach. But as mentioned in the introduction, interesting solution can be found and we hope that the results presented in this article makes finding them easier. An interesting sub-case to consider would be a theory with an $\mathrm{SU}(2)$ Fayet-Iliopoulos term along the lines of ref. [39]; work in this direction is in progress.

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A Pauli matrices

The Hermitean, unitary, traceless, 2×2 Pauli matrices σ^x ($x = 1, 2, 3$) are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{A.1}$$

They satisfy the following properties:

$$(\sigma^x)^I{}_J(\sigma^y)^J{}_K = \delta^{xy}\delta^I{}_K + i\varepsilon^{xyz}(\sigma^z)^I{}_K\,, \tag{A.2}$$

$$\delta^K{}_J\delta^L{}_I = \tfrac{1}{2}\delta^K{}_I\delta^L{}_J + \tfrac{1}{2}(\sigma^m)^K{}_I(\sigma^m)^L{}_J\,, \tag{A.3}$$

$$\varepsilon^{IJ}\varepsilon_{KL} = -\tfrac{2}{3}(\sigma^x)^{[I}{}_{[K}(\sigma^x)^{J]}{}_{L]}\,, \tag{A.4}$$

$$(\sigma^{[x|})^I{}_J(\sigma^{|y]})^K{}_L = -\tfrac{i}{2}\varepsilon^{xyz}[\delta^I{}_L(\sigma^z)^K{}_J - (\sigma^z)^I{}_L\delta^K{}_J]\,, \tag{A.5}$$

$$\varepsilon_{K[I}(\sigma^x)^K{}_{J]} = (\sigma^x)^{[I}{}_K\varepsilon^{J]K} = 0\,, \tag{A.6}$$

$$\varepsilon_{LI}(\sigma^x)^I{}_J\varepsilon^{JK} = (\sigma^x)^K{}_L\,, \tag{A.7}$$

$$\left[(\sigma^x)^I{}_J\varepsilon^{JK}\right]^* = -\varepsilon_{IJ}(\sigma^x)^J{}_K\,. \tag{A.8}$$

Taking into account the above properties, we have the following decompositions:

$$\begin{aligned} V^I{}_J &= \tfrac{1}{2}V\,\delta^I{}_J + \tfrac{1}{\sqrt{2}}\,V^x\,(\sigma^x)^I{}_J\,, \\ V &= \delta^J{}_I V^I{}_J\,, \\ V^x{}_\mu &= \tfrac{1}{\sqrt{2}}(\sigma^x)^J{}_I V^I{}_J\,, \end{aligned} \tag{A.9}$$

where V and V^x are real if $V^I{}_J$ is Hermitian, and

$$\begin{aligned} A_{IJ} &= \tfrac{1}{2}A\,\varepsilon_{IJ} + \tfrac{i}{\sqrt{2}}A^x\,\varepsilon_{IK}(\sigma^x)^K{}_J\,, \\ A &= \varepsilon^{IJ}A_{IJ}\,, \\ A^x &= \tfrac{i}{\sqrt{2}}(\sigma^x)^I{}_K\varepsilon^{KJ}A_{IJ}\,, \end{aligned} \tag{A.10}$$

or

$$\begin{aligned} A^{IJ} &\equiv (A_{IJ})^* = \tfrac{1}{2}A^*\,\varepsilon^{IJ} + \tfrac{i}{\sqrt{2}}A^{x\,*}\,(\sigma^x)^I{}_K\varepsilon^{KJ}\,, \\ A^* &= \varepsilon_{IJ}A^{IJ}\,, \\ A^{x\,*} &= \tfrac{i}{\sqrt{2}}\varepsilon_{IK}(\sigma^x)^K{}_J A^{IJ}\,. \end{aligned} \tag{A.11}$$

In the particular case of the 1- and 2-form spinor bilinears $\hat{V}^I{}_J$ and $\hat{\Phi}_{IJ}$ these decompositions are related by [32]

$$\hat{\Phi}^x = \frac{i}{2X^*}\left[\hat{V}^x \wedge \hat{V} + i\star(\hat{V}^x \wedge \hat{V})\right]\,. \tag{A.12}$$

B Gauging holomorphic isometries of Special Kähler Geometries

In this appendix we will review some basics of the gauging of holomorphic isometries of the special Kähler manifold in $N = 2$, $d = 4$ supergravities coupled to vector supermultiplets with the aim of fixing our conventions.

We start by assuming that the Hermitean metric $\mathcal{G}_{ij^*}$ admits a set of Killing vectors[12] $\{K_\Lambda = k_\Lambda{}^i\partial_i + k^*_\Lambda{}^{i^*}\partial_{i^*}\}$ satisfying the Lie algebra

$$[K_\Lambda, K_\Sigma] = -f_{\Lambda\Sigma}{}^\Omega K_\Omega\,, \tag{B.1}$$

of the group G_V that we want to gauge.

Hermiticity and the ij and i^*j^* components of the Killing equation imply that the components $k_\Lambda{}^i$ and $k^*_\Lambda{}^{i^*}$ of the Killing vectors are, respectively, holomorphic and anti-holomorphic and satisfy, separately, the above Lie algebra. Once (anti-) holomorphicity is taken into account, the only non-trivial components of the Killing equation are

$$\tfrac{1}{2}\pounds_\Lambda \mathcal{G}_{ij^*} = \nabla_{i^*}k^*_{\Lambda\,j} + \nabla_j k_{\Lambda\,i^*} = 0\,, \tag{B.2}$$

where $\pounds_\Lambda$ stands for the Lie derivative w.r.t. K_Λ.

The standard σ-model kinetic term $\mathcal{G}_{ij^*}\partial_\mu Z^i\partial^\mu Z^{*j^*}$ is automatically invariant under infinitesimal reparametrizations of the form

$$\delta_\alpha Z^i = \alpha^\Lambda k_\Lambda{}^i\,, \tag{B.3}$$

if the α^Λs are constants. If they are arbitrary functions of the spacetime coordinates $\alpha^\Lambda(x)$ we need to introduce a covariant derivative using as connection the vector fields present in the theory. The covariant derivative is

$$\mathfrak{D}_\mu Z^i = \partial_\mu Z^i + gA^\Lambda{}_\mu k_\Lambda{}^i\,, \tag{B.4}$$

and transforms as

$$\delta_\alpha \mathfrak{D}_\mu Z^i = \alpha^\Lambda(x)\partial_j k_\Lambda{}^i \mathfrak{D}_\mu Z^j = -\alpha^\Lambda(x)(\pounds_\Lambda - K_\Lambda)\mathfrak{D}_\mu Z^j\,, \tag{B.5}$$

provided that the gauge potentials transform as

$$\delta_\alpha A^\Lambda{}_\mu = -g^{-1}\mathfrak{D}_\mu\alpha^\Lambda \equiv -g^{-1}(\partial_\mu\alpha^\Lambda + gf_{\Sigma\Omega}{}^\Lambda A^\Sigma{}_\mu\alpha^\Omega)\,. \tag{B.6}$$

For any tensor[13] Φ transforming covariantly under gauge transformations, i.e. tranforming as

$$\delta_\alpha\Phi = -\alpha^\Lambda(x)(\pounds_\Lambda - K_\Lambda)\Phi\,, \tag{B.7}$$

the gauge covariant derivative is given by

$$\mathfrak{D}_\mu\Phi = \{\nabla_\mu + \mathfrak{D}_\mu Z^i\Gamma_i + \mathfrak{D}_\mu Z^{*i^*}\Gamma_{i^*} - gA^\Lambda{}_\mu(\pounds_\Lambda - K_\Lambda)\}\Phi\,. \tag{B.8}$$

In particular, on $\mathfrak{D}_\mu Z^i$

[12] The index Λ always takes values from 1 to $\bar{n}$, but some (or all) the Killing vectors may be zero.

[13] Spacetime and target space tensor indices are not explicitly shown.

$$\mathfrak{D}_\mu \mathfrak{D}_\nu Z^i = \nabla_\mu \mathfrak{D}_\nu Z^i + \Gamma_{jk}{}^i \mathfrak{D}_\mu Z^j \mathfrak{D}_\nu Z^k + g A^\Lambda{}_\mu \partial_j k_\Lambda{}^i \mathfrak{D}_\nu Z^j \,, \tag{B.9}$$

$$[\mathfrak{D}_\mu, \mathfrak{D}_\nu] Z^i = g F^\Lambda{}_{\mu\nu} k_\Lambda{}^i \,, \tag{B.10}$$

where

$$F^\Lambda{}_{\mu\nu} = 2\partial_{[\mu} A^\Lambda{}_{\nu]} + g f_{\Sigma\Omega}{}^\Lambda A^\Sigma{}_{[\mu} A^\Omega{}_{\nu]} \,, \tag{B.11}$$

is the gauge field strength and transforms under gauge transformations as

$$\delta_\alpha F^\Lambda{}_{\mu\nu} = -\alpha^\Sigma(x) f_{\Sigma\Omega}{}^\Lambda \, F^\Omega{}_{\mu\nu} \,. \tag{B.12}$$

An important case is that of tensors which only depend on the spacetime coordinates through the complex scalars Z^i and their complex conjugates so that $\nabla_\mu \Phi = \partial_\mu \Phi = \partial_\mu Z^i \partial_i \Phi + \partial_\mu Z^{*i^*} \partial_{i^*} \Phi$. This can only be true irrespectively of gauge transformations if the tensor Φ is invariant, that is

$$\pounds_\Lambda \Phi = 0 \,. \tag{B.13}$$

The gauge covariant derivative of invariant tensors is always the covariant pullback of the target covariant derivative:

$$\mathfrak{D}_\mu \Phi = \mathfrak{D}_\mu Z^i \nabla_i \Phi + \mathfrak{D}_\mu Z^{*i^*} \nabla_{i^*} \Phi \,. \tag{B.14}$$

Now, to make the σ-model kinetic gauge invariant it is enough to replace the partial derivatives by covariant derivatives.

In $N = 2$, $d = 4$ supergravity, however, the scalar manifold is not just Hermitean, but special Kähler, and simple isometries of the metric are not necessarily symmetries of the theory: they must respect the special Kähler structure. Let us first study how the Kähler structure is preserved.

The transformations generated by the Killing vectors will preserve the Kähler structure if they leave the Kähler potential invariant up to Kähler transformations, i.e., for each Killing vector K_Λ

$$\pounds_\Lambda \mathcal{K} \equiv k_\Lambda{}^i \partial_i \mathcal{K} + k^*_\Lambda{}^{i^*} \partial_{i^*} \mathcal{K} = \lambda_\Lambda(Z) + \lambda^*_\Lambda(Z^*) \,. \tag{B.15}$$

From this condition it follows that

$$\pounds_\Lambda \lambda_\Sigma - \pounds_\Sigma \lambda_\Lambda = -f_{\Lambda\Sigma}{}^\Omega \lambda_\Omega \,. \tag{B.16}$$

On the other hand, the preservation of the Kähler structure implies the conservation of the Kähler 2-form $\mathcal{J}$

$$\pounds_\Lambda \mathcal{J} = 0 \,. \tag{B.17}$$

The closedness of $\mathcal{J}$ implies that $\pounds_\Lambda \mathcal{J} = d(i_{k_\Lambda} \mathcal{J})$ and therefore the preservation of the Kähler structure implies the existence of a set of real 0-forms $\mathcal{P}_\Lambda$ known as *momentum map* such that

$$i_{k_\Lambda} \mathcal{J} = \mathcal{P}_\Lambda \,. \tag{B.18}$$

A local solution for this equation is provided by

$$i\mathcal{P}_\Lambda = k_\Lambda{}^i\partial_i\mathcal{K} - \lambda_\Lambda\,, \tag{B.19}$$

which, on account of eq. (B.15) is equivalent to

$$i\mathcal{P}_\Lambda = -(k^*_\Lambda{}^{i^*}\partial_{i^*}\mathcal{K} - \lambda^*_\Lambda)\,, \tag{B.20}$$

or

$$\mathcal{P}_\Lambda = i_{k_\Lambda}\mathcal{Q} - \tfrac{1}{2i}(\lambda_\Lambda - \lambda^*_\Lambda)\,. \tag{B.21}$$

The momentum map can be used as a prepotential from which the Killing vectors can be derived:

$$k_{\Lambda\, i^*} = i\partial_{i^*}\mathcal{P}_\Lambda\,. \tag{B.22}$$

Using eqs. (B.1),(B.15) and (B.16) one finds

$$\pounds_\Lambda \mathcal{P}_\Sigma = 2ik_{[\Lambda}{}^i k^*_{\Sigma]}{}^{j^*}\mathcal{G}_{ij^*} = -f_{\Lambda\Sigma}{}^\Omega\mathcal{P}_\Omega\,. \tag{B.23}$$

The gauge transformation rule a symplectic section Φ of Kähler weight (p,q) is[14]

$$\delta_\alpha\Phi = -\alpha^\Lambda(x)(\mathbb{L}_\Lambda - K_\Lambda)\Phi\,, \tag{B.24}$$

where $\mathbb{L}_\Lambda$ stands for the symplectic and Kähler-covariant Lie derivative w.r.t. K_Λ and is given by

$$\mathbb{L}_\Lambda\Phi = \{\pounds_\Lambda - [\mathcal{S}_\Lambda - \tfrac{1}{2}(p\lambda_\Lambda + q\lambda^*_\Lambda)]\}\Phi\,, \tag{B.25}$$

where the $\mathcal{S}_\Lambda$ are $\mathfrak{sp}(\bar{n},\mathbb{R})$ matrices that provide a representation of the Lie algebra of the gauge group G_V:

$$[\mathcal{S}_\Lambda, \mathcal{S}_\Sigma] = +f_{\Lambda\Sigma}{}^\Omega\mathcal{S}_\Omega\,. \tag{B.26}$$

The gauge covariant derivative acting on these sections is given by

$$\begin{aligned}\mathfrak{D}_\mu\Phi \;\; = \;\; & \{\nabla_\mu + \mathfrak{D}_\mu Z^i\Gamma_i + \mathfrak{D}_\mu Z^{*i^*}\Gamma_{i^*} + \tfrac{1}{2}(pk_\Lambda{}^i\partial_i\mathcal{K} + qk^*_\Lambda{}^{i^*}\partial_{i^*}\mathcal{K}) \\ & + gA^\Lambda{}_\mu[\mathcal{S}_\Lambda + \tfrac{i}{2}(p-q)\mathcal{P}_\Lambda - (\pounds_\Lambda - K_\Lambda)]\}\Phi\,.\end{aligned} \tag{B.27}$$

Invariant sections are those for which

$$\mathbb{L}_\Lambda\Phi = 0\,, \quad \Rightarrow \quad \pounds_\Lambda\Phi = [\mathcal{S}_\Lambda - \tfrac{1}{2}(p\lambda_\Lambda + q\lambda^*_\Lambda)]\Phi\,, \tag{B.28}$$

and their gauge covariant derivatives are, again, the covariant pullbacks of the Kähler-covariant derivatives:

$$\mathfrak{D}_\mu\Phi = \mathfrak{D}_\mu Z^i\mathcal{D}_i\Phi + \mathfrak{D}_\mu Z^{*i^*}\mathcal{D}_{i^*}\Phi\,. \tag{B.29}$$

By hypothesis (preservation of the special Kähler structure), the canonical weight $(1,-1)$ section $\mathcal{V}$ is an invariant section

$$K_\Lambda\mathcal{V} = [\mathcal{S}_\Lambda - \tfrac{1}{2}(\lambda_\Lambda - \lambda^*_\Lambda)]\mathcal{V}\,, \tag{B.30}$$

and its gauge covariant derivative is given by

[14]Again, spacetime and target space tensor indices are not explicitly shown. Symplectic indices are not shown, either.

$$\mathfrak{D}_\mu \mathcal{V} = \mathfrak{D}_\mu Z^i \mathcal{D}_i \mathcal{V} = \mathfrak{D}_\mu Z^i \mathcal{U}_i \,. \tag{B.31}$$

Using the covariant holomorphicity of $\mathcal{V}$ one can write

$$K_\Lambda \mathcal{V} = k_\Lambda{}^i \mathcal{U}_i - i\mathcal{P}_\Lambda \mathcal{V} - \tfrac{1}{2}(\lambda_\Lambda - \lambda^*_\Lambda)\mathcal{V} \,, \tag{B.32}$$

and, comparing with eq. (B.30) and taking the symplectic product with $\mathcal{V}^*$, we find another expression for the momentum map

$$\mathcal{P}_\Lambda = \langle\, \mathcal{V}^* \mid \mathcal{S}_\Lambda \mathcal{V} \,\rangle \,, \tag{B.33}$$

which leads, via eq. (B.22) to another expression for the Killing vectors

$$k_\Lambda{}^i = i\partial^i \mathcal{P}_\Lambda = i\langle\, \mathcal{V} \mid \mathcal{S}_\Lambda \mathcal{U}^{*i} \,\rangle \,. \tag{B.34}$$

If we take the symplectic product with $\mathcal{V}$ instead, we get the following condition

$$\langle\, \mathcal{V} \mid \mathcal{S}_\Lambda \mathcal{V} \,\rangle = 0 \,. \tag{B.35}$$

Using the same identity and $\mathcal{G}_{ij^*} = -i\langle \mathcal{U}_i \mid \mathcal{U}^*_{j^*} \,\rangle$ one can also show that

$$k_\Lambda{}^i k^*_\Sigma{}^{j^*} \mathcal{G}_{ij^*} = \mathcal{P}_\Lambda \mathcal{P}_\Sigma - i\langle\, \mathcal{S}_\Lambda \mathcal{V} \mid \mathcal{S}_\Sigma \mathcal{V}^* \,\rangle \,. \tag{B.36}$$

It follows that

$$\langle\, \mathcal{S}_{[\Lambda} \mathcal{V} \mid \mathcal{S}_{\Sigma]} \mathcal{V}^* \,\rangle \; = -\tfrac{1}{2} f_{\Lambda\Sigma}{}^\Omega \mathcal{P}_\Omega . \tag{B.37}$$

The gauge covariant derivative of $\mathcal{U}_i$ is

$$\mathfrak{D}_\mu \mathcal{U}_i = \mathfrak{D}_\mu Z^j \mathcal{D}_j \mathcal{U}_i + \mathfrak{D}_\mu Z^{*j^*} \mathcal{D}_{j^*} \mathcal{U}_i = i\mathcal{C}_{ijk} \mathcal{U}^{*j} \mathfrak{D}_\mu Z^k + \mathcal{G}_{ij^*} \mathcal{V} \mathfrak{D}_\mu Z^{*j^*} \,. \tag{B.38}$$

Then, explicitly, the covariant derivative on the upper $\mathcal{L}^\Lambda$ and lower $\mathcal{M}_\Lambda$ components of the canonical section and on the supersymmetry parameters ϵ_I, which are of $(\frac{1}{2}, -\frac{1}{2})$ weight, are given by

$$\mathfrak{D}_\mu \mathcal{L}^\Lambda = \partial_\mu \mathcal{L}^\Lambda + i\hat{\mathcal{Q}}_\mu \mathcal{L}^\Lambda + gA^\Omega f_{\Omega\Sigma}{}^\Lambda \mathcal{L}^\Sigma \,, \tag{B.39}$$

$$\mathfrak{D}_\mu \mathcal{M}_\Lambda = \partial_\mu \mathcal{M}_\Lambda + i\hat{\mathcal{Q}}_\mu \mathcal{M}_\Lambda - gA^\Omega f_{\Omega\Lambda}{}^\Sigma \mathcal{M}_\Sigma \,, \tag{B.40}$$

$$\mathfrak{D}_\mu \epsilon_I = \left\{ \nabla_\mu + \tfrac{i}{2} \hat{\mathcal{Q}}_\mu \right\} \epsilon_I \,, \tag{B.41}$$

where we have defined

$$\hat{\mathcal{Q}}_\mu \equiv \mathcal{Q}_\mu + gA^\Lambda{}_\mu \mathcal{P}_\Lambda \,. \tag{B.42}$$

The formalism, so far, applies to any group G_V of isometries. However, we will restrict ourselves to those for which the matrices

$$\mathcal{S}_\Lambda = \begin{pmatrix} a_\Lambda{}^\Omega{}_\Sigma & b_\Lambda{}^{\Omega\Sigma} \\ c_{\Lambda\Omega\Sigma} & d_{\Lambda\Omega}{}^\Sigma \end{pmatrix} \,, \tag{B.43}$$

have $b = c = 0$. The symplectic transformations with $b \neq 0$ are not symmetries of the action and the gauging of symmetries with $c \neq 0$ leads to the presence of complicated Chern-Simons terms in the action. The matrices a and d are

$$a_{\Lambda}{}^{\Omega}{}_{\Sigma} = f_{\Lambda\Sigma}{}^{\Omega}\,, \qquad d_{\Lambda\Omega}{}^{\Sigma} = -f_{\Lambda\Omega}{}^{\Sigma}\,. \tag{B.44}$$

These restrictions lead to additional identities. First, observe that the condition eq. (B.35) takes the form

$$f_{\Lambda\Sigma}{}^{\Omega}\mathcal{L}^{\Sigma}\mathcal{M}_{\Omega} = 0\,, \tag{B.45}$$

and the covariant derivative of eq. (B.35) $\langle\, \mathcal{V} \mid \mathcal{S}_{\Lambda}\mathcal{U}_i\,\rangle = 0$

$$f_{\Lambda\Sigma}{}^{\Omega}(f^{\Sigma}{}_{i}\mathcal{M}_{\Omega} + h_{\Omega\,i}\mathcal{L}^{\Sigma}) = 0\,. \tag{B.46}$$

Then, using eqs. (B.33) and (B.34) and eqs. (B.35),(B.45) and (B.46) we find that

$$\mathcal{L}^{\Lambda}\mathcal{P}_{\Lambda} \;=\; 0\,, \tag{B.47}$$

$$\mathcal{L}^{\Lambda}k_{\Lambda}{}^{i} \;=\; 0\,, \tag{B.48}$$

$$\mathcal{L}^{*\Lambda}k_{\Lambda}{}^{i} \;=\; -if^{*\Lambda\,i}\mathcal{P}_{\Lambda}\,. \tag{B.49}$$

From the first two equations it follows that

$$\mathcal{L}^{\Lambda}\lambda_{\Lambda} = 0\,. \tag{B.50}$$

Some further equations that can be derived and are extensively used in the calculation throughout the text are explicit versions of Eqs. (B.33) and (B.34), *i.e.*

$$\mathcal{P}_{\Lambda} \;=\; 2f_{\Lambda\Sigma}{}^{\Gamma}\Re\mathrm{e}\left(\mathcal{L}^{\Sigma}\mathcal{M}^{*}_{\Gamma}\right)\,, \qquad k_{\Lambda\,i^*} \;=\; if_{\Lambda\Sigma}{}^{\Gamma}\left(f^{*\Sigma}_{i^*}M_{\Gamma} + \mathcal{L}^{\Sigma}h^{*}_{\Gamma i^*}\right)\,. \tag{B.51}$$

Finally, notice the identity

$$k_{\Lambda\,i^*}\mathfrak{D}Z^{*i^*} - k^{*}_{\Lambda i}\mathfrak{D}Z^{i} = i\mathfrak{D}\mathcal{P}_{\Lambda} = i(d\mathcal{P}_{\Lambda} + gf_{\Lambda\Sigma}{}^{\Omega}A^{\Sigma}\mathcal{P}_{\Omega})\,. \tag{B.52}$$

The absolutely last comment in this appendix is the following: if we start from the existence of a prepotential $\mathcal{F}(\mathcal{X})$, then eq. (B.35) implies

$$0 \;=\; f_{\Lambda\Sigma}{}^{\Gamma}\,\mathcal{X}^{\Sigma}\partial_{\Gamma}\,\mathcal{F}\,, \tag{B.53}$$

the meaning of which is that one can gauge only the invariances of the prepotential. To put it differently: if you want to construct a model having $\mathfrak{g}$ as the gauge algebra, you need to pick a prepotential that is $\mathfrak{g}$-invariant.

C Gauging isometries of quaternionic Kähler manifolds

We start by assuming that the metric H_{uv} admits Killing vectors $\mathsf{k}_\Lambda{}^u$ satisfying the Lie algebra

$$[k_\Lambda, k_\Sigma] = -f_{\Lambda\Sigma}{}^\Omega k_\Omega\,, \tag{C.1}$$

where, as in previous cases, for certain values of Λ the vectors and the structure constants can vanish. The metric and the ungauged sigma model are invariant under the global transformations

$$\delta_\alpha q^u = \alpha^\Lambda \mathsf{k}_\Lambda{}^u(q)\,. \tag{C.2}$$

In order to make this global invariance local, we just have to replace the standard derivatives of the scalars by the covariant derivatives

$$\mathfrak{D}_\mu q^u \equiv \partial_\mu q^u + gA^\Lambda{}_\mu \mathsf{k}_\Lambda{}^u\,, \tag{C.3}$$

which will transform according to

$$\delta_\alpha \mathfrak{D}_\mu q^u = \alpha^\Lambda(x)\partial_v \mathsf{k}_\Lambda{}^u \mathfrak{D}_\mu q^v\,, \tag{C.4}$$

provided that the gauge potentials transform in the standard form eq. (B.6).

This is enough to gauge the global symmetry of the scalars' kinetic term. However, the isometries of the metric need not be global symmetries of the full supergravity theory. They have to preserve the quaternionic-Kähler structure as well, and not just the metric. In order to discuss the preservation of this structure, we need to define $\mathrm{SU}(2)$-covariant Lie derivatives.

Let $\psi^x(q)$ be a field on HM transforming under infinitesimal local $\mathrm{SU}(2)$ transformations according to

$$\delta_\lambda \psi^x = -\varepsilon^{xyz}\lambda^y\psi^z\,. \tag{C.5}$$

Its $\mathrm{SU}(2)$-covariant derivative is given by

$$\mathsf{D}\psi^x = d\psi^x + \varepsilon^{xyz}\mathsf{A}^y\psi^z\,, \tag{C.6}$$

where the $\mathrm{SU}(2)$-connection 1-form transforms as

$$\delta_\lambda \mathsf{A}^x = \mathsf{D}\lambda^x\,. \tag{C.7}$$

To define an $\mathrm{SU}(2)$-covariant Lie derivative with respect to the Killing vector k_Λ $\mathbb{L}_\Lambda$, we add to the standard one $\pounds_\Lambda$ a local $\mathrm{SU}(2)$ transformation whose transformation parameter is given by the compensator field $\mathsf{W}_\Lambda{}^x$:

$$\mathbb{L}_\Lambda\psi^x \equiv \pounds_\Lambda\psi^x + \varepsilon^{xyz}\mathsf{W}_\Lambda{}^y\psi^z\,, \tag{C.8}$$

which is such that

$$\delta_\lambda \mathsf{W}_\Lambda{}^x = \pounds_\Lambda\lambda^x - \varepsilon^{xyz}\lambda^y\mathsf{W}_\Lambda{}^z = \mathbb{L}_\Lambda\lambda^x\,. \tag{C.9}$$

$\mathbb{L}_\Lambda$ is clearly a linear operator which satisfies the Leibnitz rule for scalar and vector products of $\mathrm{SU}(2)$ vectors. The Lie derivative must also satisfy

$$[\mathbb{L}_\Lambda,\, \mathbb{L}_\Sigma] = \mathbb{L}_{[\mathsf{k}_\Lambda,\,\mathsf{k}_\Sigma]} \tag{C.10}$$

which implies the Jacobi identity. This requires

$$\pounds_\Lambda \mathsf{W}_\Sigma{}^x - \pounds_\Sigma \mathsf{W}_\Lambda{}^x + \varepsilon^{xyz}\mathsf{W}_\Lambda{}^y \mathsf{W}_\Sigma^z = -f_{\Lambda\Sigma}{}^\Gamma \mathsf{W}_\Gamma{}^x\,, \tag{C.11}$$

where, due to the assumed linear dependency of W_Λ on k_Λ, $\mathsf{W}_{[\mathsf{k}_\Lambda,\,\mathsf{k}_\Sigma]} = -f_{\Lambda\Sigma}{}^\Gamma \mathsf{W}_\Gamma$.

In order to satisfy equation (C.11) we introduce another $\mathrm{SU}(2)$ vector $\mathsf{P}_\Lambda{}^x$ such that

$$\mathsf{W}_\Lambda{}^x \equiv \mathsf{k}_\Lambda{}^u \mathsf{A}^x{}_u - \mathsf{P}_\Lambda{}^x\,, \tag{C.12}$$

which has to satisfy the equivariance condition

$$\mathsf{D}_\Lambda \mathsf{P}_\Sigma{}^x - \mathsf{D}_\Sigma \mathsf{P}_\Lambda{}^x - \varepsilon^{xyz}\mathsf{P}_\Lambda{}^y \mathsf{P}_\Sigma{}^z - \varkappa\, \mathsf{k}_\Lambda{}^u \mathsf{k}_\Sigma{}^v\, \mathsf{K}^x{}_{uv} = -f_{\Lambda\Sigma}{}^\Gamma \mathsf{P}_\Gamma{}^x\,, \tag{C.13}$$

where $\mathsf{D}_\Lambda \equiv \mathsf{k}_\Lambda{}^u \mathsf{D}_u$ and we have used the fundamental property of the hyperKähler manifolds

$$\mathsf{F}^x \;=\; \varkappa\, \mathsf{K}^x\,, \tag{C.14}$$

where

$$\mathsf{F}^x \;\equiv\; d\mathsf{A}^x \;+\; \tfrac{1}{2}\varepsilon^{xyz}\, \mathsf{A}^y \wedge \mathsf{A}^z\,, \tag{C.15}$$

is the field strength of the $\mathrm{SU}(2)$-connection and $\varkappa$ is a non-vanishing real number which has to be negative for the kinetic energy of the hyperscalars to be positive; we take $\varkappa = -2$ as to have a conventionally defined kinetic term for the hyperscalars.

$\mathsf{P}_\Lambda{}^x$ is going to be the *triholomorphic momentum map* when we impose the preservation of the hyperKähler structure K^x by the global transformations eq. (C.2) and the compensating $\mathrm{SU}(2)$ transformation with parameter W_Λ. This condition is expressed using $\mathbb{L}$ as

$$\mathbb{L}_\Lambda \mathsf{K}^x{}_{uv} = \pounds_\Lambda \mathsf{K}^x{}_{uv} + \varepsilon^{xyz}(\mathsf{k}_\Lambda{}^w \mathsf{A}^y{}_w - \mathsf{P}_\Lambda{}^y)\mathsf{K}^z{}_{uv} = -2\mathsf{D}_{[u|}(\mathsf{k}_\Lambda{}^w \mathsf{K}^x{}_{w|v]}) - \varepsilon^{xyz}\mathsf{P}_\Lambda{}^y \mathsf{K}^z{}_{uv} = 0\,. \tag{C.16}$$

Using the covariant constancy of the hyperKähler structure, this condition can be rewritten in the form

$$2(\nabla_{[u|}\mathsf{k}_\Lambda{}^w)\mathsf{K}^x{}_{w|v]} - \varepsilon^{xyz}\mathsf{P}_\Lambda{}^y \mathsf{K}^z{}_{uv} = 0\,, \tag{C.17}$$

and, contracting the whole equation with $\mathsf{K}^{y\,uv}$ we find

$$\mathsf{K}^{x\,uv}\nabla_u \mathsf{k}_{\Lambda\,v} = -2m\mathsf{P}_\Lambda{}^x\,. \tag{C.18}$$

Acting on both sides of this equations with D_w and using the Killing vector identity $\nabla_w\nabla_u \mathsf{k}_{\Lambda\,v} = R_{wruv}\mathsf{k}_\Lambda{}^r$ we get

$$\mathsf{k}_\Lambda{}^r R_{wruv}\mathsf{K}^{x\,uv} = -2m\mathsf{D}_w \mathsf{P}_\Lambda{}^x\,. \tag{C.19}$$

Finally, using

$$\mathsf{K}^x{}_{uv} = -i\sigma^x{}_{IJ}\mathsf{U}^{\alpha I}{}_u \mathsf{U}^{\beta J}{}_v \mathbb{C}_{\alpha\beta}\,, \qquad \sigma^x{}_{IJ} \equiv \sigma^x{}_I{}^K \varepsilon_{JK}\,, \tag{C.20}$$

in

$$R_{ts}{}^{uv}\,\mathsf{U}^{\alpha I}{}_u\,\mathsf{U}^{\beta J}{}_v \;=\; -\mathsf{G}_{ts}^{IJ}\,\mathbb{C}^{\alpha\beta} \;-\; \overline{\mathsf{R}}_{ts}^{\alpha\beta}\,\varepsilon^{IJ} \;=\; \mathsf{F}_{ts}^{IJ}\,\mathbb{C}^{\alpha\beta} \;-\; \overline{\mathsf{R}}_{ts}^{\alpha\beta}\,\varepsilon^{IJ}\,, \tag{C.21}$$

we get

$$R_{wruv}\mathsf{K}^{x\,uv} = -2m\,\mathsf{F}^x{}_{wr} = -2m\varkappa\,\mathsf{K}^x{}_{wr}\,. \tag{C.22}$$

Substituting above, we arrive at

$$\mathsf{D}_u\mathsf{P}_\Lambda{}^x = \varkappa\,\mathsf{K}^x{}_{uv}\mathsf{k}_\Lambda{}^v\,, \tag{C.23}$$

which can be taken as the defining equation of the triholomorphic momentum map. From this equation we find

$$\mathsf{D}_\Sigma\mathsf{P}_\Lambda{}^x = \varkappa\,\mathsf{k}_\Sigma{}^u\mathsf{k}_\Lambda{}^v\mathsf{K}^x{}_{uv}\,, \tag{C.24}$$

and, substituting directly in eq. (C.13) we obtain

$$\mathbb{L}_\Lambda\mathsf{P}_\Sigma{}^x = \mathsf{D}_\Lambda\mathsf{P}_\Sigma{}^x - \varepsilon^{xyz}\mathsf{P}_\Lambda{}^y\mathsf{P}_\Sigma{}^z + f_{\Lambda\Sigma}{}^\Omega\mathsf{P}_\Omega{}^x = 0\,, \tag{C.25}$$

which says that the triholomorphic momentum map is an invariant field and

$$\varepsilon^{xyz}\mathsf{P}_\Lambda{}^y\mathsf{P}_\Sigma{}^z - \varkappa\,\mathsf{k}_\Lambda{}^u\mathsf{k}_\Sigma{}^v\mathsf{K}^x{}_{uv} = f_{\Lambda\Sigma}{}^\Omega\mathsf{P}_\Omega{}^x\,. \tag{C.26}$$

Now, for a field Φ (possibly with spacetime, quaternionic, $\mathrm{SU}(2)$ or gauge indices) which under eq. (C.2) transforms according to

$$\delta_\alpha\Phi = -\alpha(\mathbb{L}_\Lambda - \mathsf{k}_\Lambda)\Phi\,, \tag{C.27}$$

we define the gauge covariant derivative

$$\mathfrak{D}_\mu\Phi \equiv \{\nabla_\mu + \mathfrak{D}_\mu q^u\Gamma_u - gA^\Lambda{}_\mu(\mathbb{L}_\Lambda - \mathsf{k}_\Lambda) + \mathfrak{D}_\mu q^u\mathsf{A}^x{}_u\}\Phi\,. \tag{C.28}$$

For the triholomorphic momentum map, we have, on account of eq. (C.25), which we can rewrite in the form

$$k_\Lambda{}^u\partial_u\mathsf{P}_\Sigma{}^x = -\varepsilon^{xyz}(k_\Lambda{}^u\mathsf{A}^y{}_u - \mathsf{P}_\Lambda{}^y)\mathsf{P}_\Sigma{}^z - f_{\Lambda\Sigma}{}^\Omega\mathsf{P}_\Omega{}^x\,, \tag{C.29}$$

the following expressions for its gauge covariant derivative

$$\mathfrak{D}_\mu\mathsf{P}_\Lambda{}^x = \partial_\mu\mathsf{P}_\Lambda{}^x + \varepsilon^{xyz}\hat{\mathsf{A}}^y{}_\mu\mathsf{P}_\Lambda{}^z + f_{\Lambda\Sigma}{}^\Omega A^\Sigma{}_\mu\mathsf{P}_\Omega{}^x\,, \tag{C.30}$$

$$\mathfrak{D}_\mu\mathsf{P}_\Lambda{}^x = \mathfrak{D}_\mu q^u\mathsf{D}_u\mathsf{P}_\Lambda{}^x\,, \tag{C.31}$$

where we have defined

$$\hat{\mathsf{A}}^x{}_\mu \equiv \partial_\mu q^u\mathsf{A}^x{}_u + gA^\Lambda{}_\mu\mathsf{P}_\Lambda{}^x\,. \tag{C.32}$$

Under eq. (C.2), spinors with $\mathrm{SU}(2)$ indices undergo the following transformation

$$\delta_\alpha\psi_I = -\alpha^\Lambda\mathsf{W}_\Lambda{}^x\tfrac{i}{2}\sigma^x{}_I{}^J\psi_J\,. \tag{C.33}$$

Then, using the general formula, their covariant derivative is given by

$$\mathfrak{D}_\mu \psi_I = \nabla_\mu \psi_I + \hat{\mathsf{A}}^x{}_\mu \, \tfrac{i}{2}\sigma^x{}_I{}^J \psi_J \, . \tag{C.34}$$

If we take into account their Kähler weight and possible gaugings of the isometries of the special-Kähler manifold, we have for the supersymmetry parameters of $N = 2$, $d = 4$ supergravity

$$\mathfrak{D}_\mu \epsilon_I = \{\nabla_\mu + \tfrac{i}{2}\hat{\mathcal{Q}}_\mu\}\epsilon_I + \hat{\mathsf{A}}^x{}_\mu \tfrac{i}{2}\sigma^x{}_I{}^J \epsilon_J \, . \tag{C.35}$$

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